MATH JUNK

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Trigonometry Stuff

Patterns for the Trigonometric Functions

I'm always amazed by how poorly many people—among them college students—know the values of trigonometric functions for the special angles. Here is one way to remember them without drawing triangles all the time.

	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	$\frac{\sqrt{0}}{2}=0$	$\frac{\sqrt{4}}{2}=1$	0*
$\pi/6 = 30°$	$\frac{\sqrt{1}}{2}=\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$3^{-1/2}$
$\pi/4 = 45°$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$3^{0/2}=1$
$\pi/3 = 60°$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2}=\frac{1}{2}$	$3^{1/2}$
$\pi/2 = 90°$	$\frac{\sqrt{4}}{2}=1$	$\frac{\sqrt{0}}{2}=0$	∞*

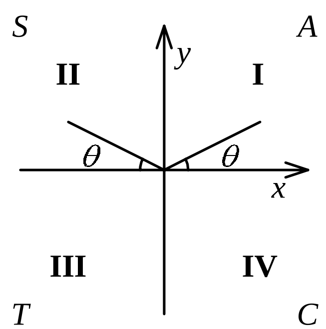
* note that tan 0 and tan 90° do not follow the pattern exhibited by the other three special angles. Instead, use the property

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

The **patterns** for sin, cos, and tan, if not already evident from the table at right:

$\sin\theta$	$\frac{\sqrt{n}}{2}$, where n ranges from 0 through 4 in correspondence to the special angles.
$\cos\theta$	same as $\sin\theta$ except in **reverse order** with respect to the special angles. (By definition, $\cos\theta = \sin(90° - \theta)$)
$\tan\theta$	$3^{n/2}$, where n ranges from –1 through +1 in correspondence with the special angles 30°, 45°, and 60°.

The Four Quadrants

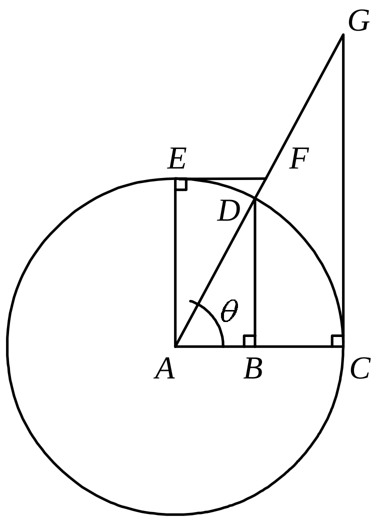


This is pretty basic. The coordinate plane is divided into four quadrants by the x- and y-axes, numbered I through IV in a counterclockwise manner, and angles are measured with respect to the x-axis. (Angles from 0–90° lie in quadrant I, 90°–180° in II, 180 –270° in III, and 270°–360° in IV.) The trigonometric functions change signs from quadrant from quadrant; all are positive in quadrant I, only sine is positive in II, only tangent in III, and only cosine in IV.

A **mnemonic device** often used to remember the signs of the trigonometric functions in each of the four quadrants: "**All students take calculus**." Each word corresponds to each of the quadrants in increasing order, and the first letter of each word relates to what is positive in that quadrant (all, sin, tan, cos). (Personally, I think "All students take classes" is better, but math teachers tend to favor the former version…)

Trigonometric Functions and the Unit Circle

These graphical representations of the trig functions are mostly useless for practical purposes (in the unlikely situation that one must evaluate a trigonometric function with a ruler and a protractor, one might as well draw the right triangle and forget about its relation to a unit circle), but it is somewhat interesting. The sine and cosine representations, however, appear often in subjects such as physics (e.g.: rotation, oscillation).



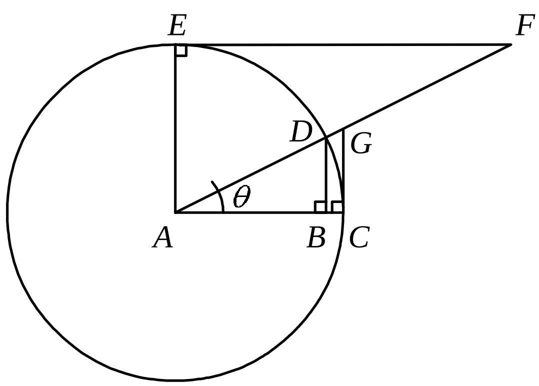
$$AC = AD = AE = \text{radius}$$

$$\frac{AB}{\text{radius}} = \cos\theta \qquad \frac{AG}{\text{radius}} = \sec\theta$$

$$\frac{BD}{\text{radius}} = \sin\theta \qquad \frac{AF}{\text{radius}} = \csc\theta$$

$$\frac{CG}{\text{radius}} = \tan\theta \qquad \frac{EF}{\text{radius}} = \cot\theta$$

These can be easily verified using the geometric properties of the trig functions (e.g.: for a right triangle, sin ≡ opposite : hypotenuse, cos ≡ adjacent : hypotenuse, etc., where "opposite," "adjacent," and "hypotenuse" represent the lengths of the respective sides) and angle congruences.



For convenience, a unit circle (radius ≡ 1) is usually considered.

Miscellaneous

A **mnemonic device** to remember the **angle addition formula for cosine**:

$$\cos(\alpha + \beta) = (\cos\alpha)(\cos\beta) - (\sin\alpha)(\sin\beta)$$

COS
- C is for **c**osines (the first term is a product of cosine functions)
- O is for **o**pposite (the operator is of opposite sign; a "+" on one side of the equation produces a "–"on the other side)
- S is for **s**ines (the second term is a product of sine functions)

Polynomial Stuff

Factoring Quadratics

Most everyone knows that a quadratic expression of the form $x^2 + bx + c$ can be factored to $(x + m)\ (x + n)$ where $m + n = b$ and $m * n = c$. Provided that b and c are not large, integral roots, if they exist, may be quickly found by inspection or with minimal work. All one needs to do is to mentally factor c until a pair of factors that sum to b is found.

Of course, oftentimes one is given a quadratic in which the leading coefficient is not unitary, and finding the roots by factoring is no longer quite as straightforward. One, however, does not necessarily need to resort to using the quadratic equation. As one might expect, **a technique to factor more general quadratic expressions** exists that requires only minor modifications to the above method.

Given a (factorable) quadratic equation:

$$a\,x^2 + b\,x + c = 0$$

1. find factors (m, n) of $a * c$ that sum to b
2. the factored form of the quadratic will then be:
$$\frac{(a\,x + m)(a\,x + n)}{a}$$
3. if rational roots to the original equation exist, the factored form can be simplified to eliminate the denominator

EXAMPLE:

factor: $15x^2 + 2x - 8$

1. $15 * -8 = -120$ (the product of a and c)
   $-120 = 12 * -10$ (factors of the above…)
   $12 + -10 = 2$ (…which sum to b)

2. the factored form of the quadratic is then:
$$\frac{(15x + 12)(15x - 10)}{15}$$

3. which can then be simplified to:
$$\frac{(3 * (5x + 4))(5 * (3x - 2))}{15} = (5x + 4)(3x - 2)$$

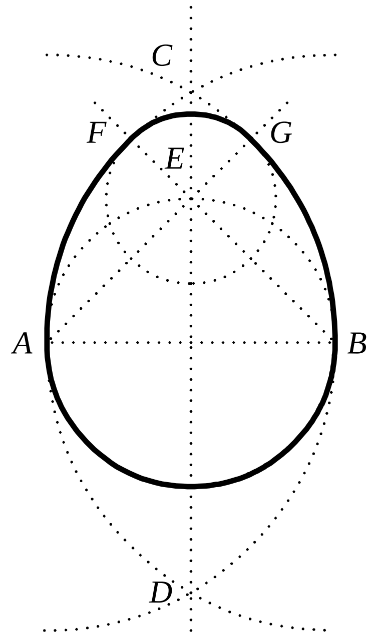
Descartes' Rule of Signs

The Fundamental Theorem of Algebra states that an n^{th}-degree polynomial has n roots, some of them possibly being degenerate. Descartes' Rule of Signs supplies **upper bounds to the numbers of positive and negative real roots**. If the terms in the polynomial $P_n(x)$ are ordered by degree, the **number of sign changes** between terms gives the maximum possible number of positive roots. The actual number must be an **even difference** less than this upper bound. Applying the same rule to $P_n(-x)$ produces the number of negative real roots.

> EXAMPLE:
>
> LET: $P_7(x) \equiv -x^7 + x^4 + x^3 - x + 1$
>
> Since there are 3 sign changes, $P_7(x)$ has either 3 or 1 positive real roots.
>
> $P_7(-x) = x^7 + x^4 - x^3 + x + 1$
>
> Since there are 2 sign changes, $P_7(x)$ has either 2 or 0 negative real roots.

Geometry Stuff

Ovals

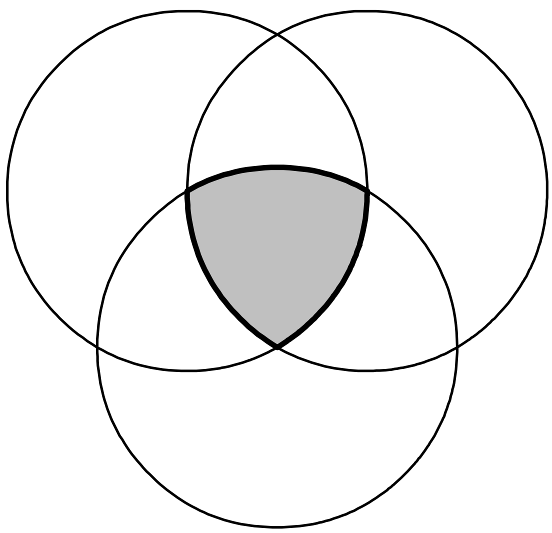


In common speech especially, the term "oval" is often used interchangeably with "ellipse." An oval, however, is not exactly the same as an ellipse; unlike an ellipse, **an oval is symmetric only about a single axis**.

One method to draw an oval using a compass and a straight-edge:

1. draw a circle and bisect it ($\overline{AB}$)
2. for each of points *A*, *B*, draw an arc centered about that point with radius set to the diameter of the circle
3. draw a line segment $\overline{CD}$ connecting the intersections of each of these arcs (thereby bisecting the circle again)
4. through point *E*, where this line intersects the circle, draw line segments $\overline{AG}$ and $\overline{BF}$
5. draw another circle centered about *E* with radius $\equiv EF = EG$

A more general method to draw ovals with different curvatures is briefly described in
The CRC Concise Encyclopedia of Mathematics
(`http://www.astro.virginia.edu/~eww6n/math/math.html`)



Reuleaux Triangles

The Reuleaux triangle is an interesting shape. Like a circle, **it is a curve of constant width**; it may be rolled between two parallel planes a fixed distance apart and always remain in contact with both. (Unlike a circle, however, its center does not remain fixed while rotating.)

A Reuleaux triangle may be constructed by drawing three circles of equal radius, centering the second circle on the edge of the first, and centering the third on an intersection of the first two.

More information on Reuleaux triangles and other curves of constant width can be found in *The CRC Concise Encyclopedia of Mathematics*.

Solid Angles

I don't know much about solid angles other than what they are. The only time I've heard of them is from my physics professor, who referred to them when describing the zero net force exerted on a charged particle from a surrounding spherical shell of charge. (A similar example would be the zero net gravitational force exerted on an object from an enclosing spherical shell.) He too mentioned that hardly any math textbooks refer to them anymore….

Solid angles, as might be surmised from their name, is a measurement of angles in three-dimensions; one might visualize a cone to represent one. More precisely, a solid angle is "defined as the surface area of a unit sphere which is subtended by a given object" and is measured in steradians (Weisstein).

Arithmetic Stuff

Properties of Logarithms

- $\log_b x^n = n \log_b x$
- $\log_b x + \log_b y = \log_b (x\,y)$
- $\log_b x - \log_b y = \log_b ({}^x/_y)$

These can be verified using the definition of logarithms.

$$\log_b x^n \stackrel{?}{=} n \log_b x$$

$$b^{\log_b x^n} \stackrel{?}{=} b^{n \log_b x}$$

$$x^n \stackrel{?}{=} (b^{\log_b x})^n$$

$$x^n = x^n \checkmark$$

$$\log_b x + \log_b y \stackrel{?}{=} \log_b (x\,y)$$

$$b^{\log_b x + \log_b y} \stackrel{?}{=} b^{\log_b (x\,y)}$$

$$b^{\log_b x} b^{\log_b y} \stackrel{?}{=} x\,y$$

$$x\,y = x\,y \checkmark$$

$$\begin{aligned} \log_b x - \log_b y &= \log_b x + \log_b y^{-1} \\ &= \log_b (x\,y^{-1}) \\ &= \log_b ({}^x/_y) \checkmark \end{aligned}$$

Change of Base with Logarithms

It is not difficult to recall the change of base formula incorrectly. It is, however, quite simple to derive if one is in doubt. Suppose one is given a number represented as $\log_a x$ but is desired in terms of logarithms of base b.

LET: $n \equiv \log_a x$

By definition of logarithms, this means that:

$a^n = x$

Taking the base-b logarithm of both sides:

$\log_b a^n = \log_b x$

$n \log_b a = \log_b x$

$$n = \frac{\log_b x}{\log_b a} = \log_a x$$

Calculus Stuff

Using Differentials to Approximate Numerical Roots

While there are algorithms to compute numerical roots to an arbitrary precision, such techniques are often somewhat complex and tedious to perform. When only an approximate value is needed, differentials may be used to evaluate roots relatively quickly.

The differential approximation method is fairly simple to understand and to implement. **It is essentially an extrapolation technique**; knowing a particular point and the behavior of the function at that point, one may guess the positions of points around it. If we wish to find points that are close to the known point, if we approximate the function's behavior to be linear, these points may be estimated with reasonable accuracy.

So to approximate roots, we first assign a function to the desired operation. In the case of square roots, for example, we may define a function $y \equiv \sqrt{x}$. We let (x, y) be a coordinate-pair that is known to be close to the desired point, (x', y'). We then differentiate this function and evaluate it at a known value to determine how far y' is from y given the distance between x' and x.

EXAMPLE:

Approximate $\sqrt{80}$

LET: $y \equiv \sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{2} x^{-1/2} \approx \frac{\Delta y}{\Delta x}$$

$$\Delta y \approx \frac{1}{2} x^{-1/2} \Delta x$$

We know that $\sqrt{80} \approx \sqrt{81} = 9$. Therefore x = 81 and $y = 9$. Thus $\Delta x = (80 - x) = -1$:

$$\Delta y \approx \frac{1}{2}(\sqrt{81})(-1) = -\frac{1}{18}$$

$$\therefore \sqrt{80} \approx y' = y + \Delta y \approx 9 - \frac{1}{18}$$

For comparison, the estimate for $\sqrt{80}$ is $8.9\overline{4}$, while the actual value is 8.94427…

Integration by Parts

Most calculus books I've seen always give the equation for integrating by parts as:

$$\int u\, dv = u\, v - \int v\, du$$

Well, as simple as this seems, in practice, I always seem to have trouble getting it straight. Halfway through the process, I often forget which factor I'm supposed to be integrating and which I'm supposed to be differentiating. A lot of people write down the separate substitutions for *u*, *v*, *du*, and *dv*, and then directly plug these expressions into the equation. This is somewhat tedious. The notation my calculus professor uses is only slightly different yet makes it a bit simpler:

$$\int u\, (v)' = u\, v - \int (u)'\, v$$

A lot of calculus texts use this prime notation when describing integration by parts, but unfortunately most use it only when presenting the equation, not in practice. **The prime notation should be used in practice**. The first step is to rewrite $\int u\, dv$ as $\int u\, (v)'$. We now know *u* and *v* without explicitly writing out the substitutions, and then we can easily determine $u\, v - \int (u)'\, v$. With this technique, we **perform the integration first** and then **clearly mark the integrated term with the prime notation**. This reduces potential confusion and allows the remaining steps to be performed without much extra thought.

EXAMPLE:

Find $\int \ln x\, dx$

$$\int \ln x\, dx = \int \ln x\ (x)' = (\ln x)\ x - \int (\ln x)'\, x = (\ln x)\ x - \int \frac{1}{x}\ x\, dx = (\ln x)\ x - \int dx = (\ln x)\ x - x + C$$

"The Hollywood Shortcut"

My high school calculus teacher showed my class he calls "the Hollywood shortcut," so dubbed because Edward James Olmos demonstrated it in the movie *Stand and Deliver*. When given an integral of the form: $\int f(x)g(x)dx$ where $f(x)$ is a polynomial expression to be reduced when integrating by parts, $f(x)$ may be successively differentiated while $g(x)$, successively integrated, until a constant derivative of $f(x)$ is reached. The resulting expressions may be multiplied together in a staggered manner and then summed with alternating positive and negative signs to generate the answer, without needing to perform the conventional integration by parts technique. (Edward James Olmos probably explains the process better in the movie, though I haven't seen it yet.) This is best described through an example:

EXAMPLE:

Find $\int x^2 e^{2x} dx$

Clearly, x^2 is the expression that will be reduced upon differentiation. We thus construct the following table:

differentiation		integration
x^2		e^{2x}
	+	
$2x$		$\frac{1}{2}e^{2x}$
	–	
2		$\frac{1}{4}e^{2x}$
	+	
0		$\frac{1}{8}e^{2x}$

We then draw lines from members of the differentiation column to those of the integration column, staggered as shown. We label each line with alternating positive and negative signs, starting with a positive one. The terms connected by the lines are multiplied together and associated with the given sign to yield the answer:

$$\int x^2 e^{2x} dx = +(x^2)(\frac{1}{2}e^{2x}) - (2x)(\frac{1}{4}e^{2x}) + (2)(\frac{1}{8}e^{2x}) + C$$

Notational Stuff

ARITHMETIC/ALGEBRAIC

$x + y$	*x* plus *y*
$x - y$	*x* minus *y*
$x \times y$	*x* times *y*
$x \cdot y$	”
$x * y$	”
$x \div y$	*x* divided by *y*
x / y	”
$x!$	factorial of *x*
$x \text{ E } y$	$x \times 10^y$
$\lceil x \rceil$	ceiling of *x*
$\lfloor x \rfloor$	floor of *x*
$x \mid y$	*y* divides *x* integrally
$(f \circ g)(x)$	compose: $f(g(x))$
$[a, b]$	range from *a* through *b* (inclusive)
(a, b)	range from *a* to *b* (exclusive)

LOGIC/PROOFS

$p \wedge q$	*p* and *q*
$p \vee q$	*p* or *q*
$p \oplus q$	*p* xor *q*
$\neg p$	not *p*
$\overline{p}$	”
$p \rightarrow q$	*p* implies *q*
$\Leftrightarrow$	truth equivalence
T	true, tautology
F	false, contradiction
$\forall x$	for all *x*
$\exists x$	there exists an *x*
$\ni$	such that
$\mid$	”
$\therefore$	therefore
$\Rightarrow$	becomes

SETS

$A \subset B$	*A* is a (proper) subset of *B*
$A \subsetneq B$	*A* is a proper subset of *B*
$A \subseteq B$	*A* subset of *B*
$a \in A$	*a* is an element of *A*
$A \cup B$	union of *A* and *B*
$A \cap B$	intersection *A* and *B*
$A \oplus B$	symmetric difference of *A* and *B*
$\varnothing$	empty/null set
R	the set of real numbers
C	the set of complex numbers
Q	the set of rational numbers
Z	the set of integers
N	the set of natural numbers
$\lvert A \rvert$	cardinality of *A*
$\#A$	”
$A \times B$	Cartesian product of *A* and *B*
$A \setminus B$	*A* less *B*
$A - B$	”
$\overline{A}$	the complement of *A*
$A \mapsto B$	mapping from *A* to *B*

GEOMETRY

$\mathbf{u} \perp \mathbf{v}$	**u** is perpendicular to **v**
$A \cong B$	*A* is congruent to *B*
$A \sim B$	*A* is similar to *B*
$\mathbf{u} \parallel \mathbf{v}$	**u** is parallel to **v**
$\mathbf{u} // \mathbf{v}$	”
$^\circ$	degrees
$'$	minutes
$''$	seconds

COMPARISONS

$x = y$	*x* equals *y*
$x \approx y$	*x* is approximately equal to *y*
$x \cong y$	”
$x \neq y$	*x* is not equal to *y*
$x \sim y$	*x* is on the order of *y*
$x \propto y$	*x* directly proportional to *y*
$x \equiv y$	*x* is set to or equivalent to *y*
$x > y$	*x* is greater than *y*
$x < y$	*x* is less than *y*
$x >> y$	*x* is much greater than *y*
$x << y$	*x* is much less than *y*
$x \equiv_n y$	*x* is congruent to *y* mod *n*

MATRICES/VECTORS

A^T	transpose of *A*
$\lvert A \rvert$	determinant of *A*
$\lVert \mathbf{u} \rVert$	norm of **u**
A^*	adjoint of *A*

CALCULUS

f'	first derivative of *f*
f''	second derivative of *f*
$f^{(n)}$	n^{th} derivative of *f*
∇f	gradient of *f*
$\dot{x}$	$\dfrac{dx}{dt}$, first derivative of *x* with respect to time

The Greek Alphabet

Α	α	alpha
Β	β	beta
Γ	γ	gamma
Δ	δ	delta
Ε	ε	epsilon
Ζ	ζ	zeta
Η	η	eta
Θ	θ	theta
Ι	ι	iota
Κ	κ	kappa
Λ	λ	lambda
Μ	μ	mu
Ν	ν	nu
Ξ	ξ	xi
Ο	ο	omicron
Π	π	pi
Ρ	ρ	rho
Σ	σ	sigma
Τ	τ	tau
Υ	υ	upsilon
Φ	φ	phi
Χ	χ	chi
Ψ	ψ	psi
Ω	ω	omega

Reference Stuff

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